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HOW TO CUT A PIZZA FAIRLY:
fair division with decreasing marginal evaluations*

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Abstract

Existential and constructive solutions to the classic problems of fair division are known for individuals with constant marginal evaluations. By considering nonatomic concave capacities instead of nonatomic probability measures, we extend some of these results to the case of individuals with decreasing marginal evaluations.

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1 Introduction

The problem of dividing an object so that everybody is satisfied with the share she receives is an ancient one. Some interesting cases are reported in the Bible (e.g., Numbers 33:54) and in the Babylonian Talmud (e.g., Kethubot 93a). Mathematical issues on the existence and construction of a solution developed at a remarkable speed after the seminal works of Steinhaus (1948) and Dubins and Spanier (1961); see, for example, Brams and Taylor (1996), Robertson and Webb (1998), and the references therein.

In the literature, the satisfaction that an individual derives from the various parts of an object S is described by a set function ν, which assigns to each part A of S the individual's (subjective) *evaluation* $\nu(A)$, with the convention $\nu(\emptyset) = 0$ and $\nu(S) = 1$. Moreover, ν is generally assumed to be additive, that is,

$$\nu(A \cup B) = \nu(A) + \nu(B)$$

if A and B are disjoint pieces.

If, for example, the individual is evaluating the various slices of a pizza, this amounts to saying that she is ***as*** *willing to eat slice A when she only ate slice B* ***as*** *when she ate both slices B and C*. In fact,

$$\nu(B \cup C \cup A) - \nu(B \cup C) = \nu(B \cup A) - \nu(B).$$

On the other hand, since pizza is quite satiating and non-storable, it is natural to investigate what happens if the individual is ***more*** *willing to eat slice A when she only ate slice B* ***rather than*** *when she ate both slices B and C*, that is, if

$$\nu(B \cup C \cup A) - \nu(B \cup C) \leq \nu(B \cup A) - \nu(B). \tag{1}$$

Set functions satisfying the above inequality for all disjoint sets A, B, and C are called concave.

The evaluation ν can take a simple form when the good to be divided is homogeneous. In this case, the individual's satisfaction depends only on the quantity $\mu(A)$ of the good she receives and it is then natural to assume that $\nu(A) = u(\mu(A))$, where u is a "utility" function. For example, if the plots of an homogeneous land are considered, $\mu(A)$ would be the Lebesgue measure of the piece of land A. It is easy to see that ν is concave provided u is concave. However, as pointed out by Shapley (1971) and Gilboa (1985), most concave evaluations do not have such a simple form.

In this article we extend to concave evaluations some classic results for additive evaluations, both existential (Section 2) and constructive (Section 3). In economics jargon, we are moving from constant to decreasing marginal utilities (see Observations 1 and 2). To the best of our knowledge, the only attempt in this direction was made by Berliant, Dunz, and Thomson (1992); their approach, however, is quite different from ours (see Section 4).

2 Preliminaries and main results

In this section we extend to the concave case the Cake Cutting and Fair Border existence theorems due to Dubins and Spanier (1961) and Hill (1983), respectively.

We first introduce some preliminary notions and properties. Given a measurable space (S, Σ), a *capacity* on Σ is a set function $\nu : \Sigma \to [0,1]$ such that:

(a) $\nu(\emptyset) = 0$ and $\nu(S) = 1$.

(b) $\nu(A) \leq \nu(B)$ for all $A, B \in \Sigma$ such that $A \subseteq B$,

(c) $\nu(A_k) \downarrow 0$ for any monotone sequence $\{A_k\} \subseteq \Sigma$ with $A_k \downarrow \emptyset$.

Assuming that the evaluation of an individual is a capacity means that she (weakly) prefers more to less and that her satisfaction decreases to zero as the received slice vanishes.

A capacity ν is *concave* (or *submodular*) if

(d) $\nu(A \cup B) \leq \nu(A) + \nu(B) - \nu(A \cap B)$ for all $A, B \in \Sigma$,

while it is *additive* (or *modular*) if

(e) $\nu(A \cup B) = \nu(A) + \nu(B)$ for all disjoint $A, B \in \Sigma$.[1]

An additive capacity is called a *probability measure*. As discussed in the introduction, condition (d) means that the marginal evaluation of the individual is decreasing, while (e) means that it is constant. In fact:

Observation 1 *Given a set function $\nu : \Sigma \to [0,1]$ such that $\nu(\emptyset) = 0$, the following statements are equivalent:*

- *ν is concave (additive, resp.);*
- *for all disjoint $A, B, C \in \Sigma$: $\nu(B \cup C \cup A) - \nu(B \cup C) \underset{(=,\ resp.)}{\leq} \nu(B \cup A) - \nu(B)$;*
- *for all $A \in \Sigma$: $\nu(B \cup A) - \nu(B)$ is decreasing (constant, resp.) in $B \in \Sigma \cap A^c$, where $\Sigma \cap A^c$ is the class of subsets of Σ disjoint from A.*

Moreover, a natural way to obtain concave set functions is to aggregate additive ones through a function with decreasing increments.

[1]Notice that this condition is equivalent to: $\nu(A \cup B) = \nu(A) + \nu(B) - \nu(A \cap B)$ for all $A, B \in \Sigma$ and $\nu(\emptyset) = 0$.

Observation 2 *Let $\mu_1, \mu_2, ..., \mu_d : \Sigma \to [0,1]$ be additive set functions and let $u : [0,1]^d \to [0,1]$ be a function such that $u(0) = 0$ and*

$$u(x+h) - u(x) \underset{(=,\ resp.)}{\leq} u(y+h) - u(y),$$

for all $x \geq y$ and all $h \geq 0$ such that $x, x+h, y, y+h \in [0,1]^d$. Then the set function

$$\nu(A) = u(\mu_1(A), \mu_2(A), ..., \mu_d(A)) \quad \forall A \in \Sigma$$

is concave (additive, resp.).[2]

Finally, a capacity ν is *nonatomic* if

(f) for each set A such that $\nu(A) > 0$ there exists $B \subseteq A$ such that $0 < \nu(B) < \nu(A)$.

It is another continuity assumption on the preferences of the individuals.

We can now state our main result, in which we consider the fair division problem with concave evaluations.

Theorem 1 (Cake Cutting) *Let $\nu_1, \nu_2, ..., \nu_n$ be nonatomic concave capacities on a measurable space (S, Σ). Given any $\alpha_1, \alpha_2, ..., \alpha_n \geq 0$ with $\sum_{i=1}^n \alpha_i = 1$, there exists a partition $\{A_1, A_2, ..., A_n\}$ of S in Σ such that*

$$\nu_i(A_i) \geq \alpha_i$$

for each $i = 1, 2, ..., n$. Moreover, if $\nu_j \neq \nu_k$ for some $j \neq k$ and $\alpha_1, \alpha_2, ..., \alpha_n > 0$, the partition $\{A_1, A_2, ..., A_n\}$ can be chosen to satisfy

$$\nu_i(A_i) > \alpha_i$$

for each $i = 1, 2, ..., n$.

Proof. Let $i \in \{1, 2, ..., n\}$ and $\mathcal{C}(\nu_i)$ be the set of all probability measures μ such that $\mu(B) \leq \nu_i(B)$ for all $B \in \Sigma$.

<u>Claim 0.</u> *For all $A \in \Sigma$, $\nu_i(A) = \max_{\mu \in \mathcal{C}(\nu_i)} \mu(A)$.*

The proof can obtained from the results of Shapley (1971) and Schmeidler (1972). In fact, the set function defined by $\bar{\nu}_i(A) = 1 - \nu_i(A^c)$ for all $A \in \Sigma$ is a "convex game continuous at S" and $\mathcal{C}(\nu_i)$ is its "core". We provide an alternative simple proof building on a result of Kelley (1959).

Let $A \in \Sigma$ and consider the probability measure λ on $\Lambda = \{\emptyset, A, A^c, S\}$ defined by $\lambda(\emptyset) = 0$, $\lambda(A) = \nu_i(A)$, $\lambda(A^c) = 1 - \nu_i(A)$, $\lambda(S) = 1$. Concavity of ν_i implies $1 - \nu_i(A) \leq \nu_i(A^c)$, hence $\lambda(L) \leq \nu_i(L)$ for all $L \in \Lambda$. By Theorem 14 of Kelley (1959), there exists an extension

[2] See, e.g., Marinacci and Montrucchio (2002).

$\hat{\lambda}$ of λ to Σ satisfying (a), (b), (e) and such that $\hat{\lambda}(B) \leq \nu_i(B)$ for every $B \in \Sigma$. Moreover, if $\{B_k\} \subseteq \Sigma$ and $B_k \downarrow \emptyset$, then $\hat{\lambda}(B_k) \leq \nu_i(B_k) \to 0$, that is $\hat{\lambda} \in \mathcal{C}(\nu_i)$. We have shown that for every $A \in \Sigma$ there exists $\hat{\lambda} \in \mathcal{C}(\nu_i)$ such that $\nu_i(A) = \hat{\lambda}(A)$; since $\nu_i(A) \geq \mu(A)$ for every $\mu \in \mathcal{C}(\nu_i)$, we have $\nu_i(A) = \max_{\mu \in \mathcal{C}(\nu_i)} \mu(A)$. $\square$

Claim 1. *There exists $\lambda_i \in \mathcal{C}(\nu_i)$ such that each $\mu \in \mathcal{C}(\nu_i)$ is absolutely continuous with respect to λ_i, that is $A \in \Sigma$ and $\lambda_i(A) = 0$ imply $\mu(A) = 0$.*

For completeness we report the proof, essentially due to Schmeidler (1972).

It is easy to see that $\mathcal{C}(\nu_i)$ is convex. If $\{A_k\} \subseteq \Sigma$ and $A_k \downarrow \emptyset$, then $\max_{\mu \in \mathcal{C}(\nu_i)} \mu(A_k) = \nu_i(A_k) \to 0$, by Theorem IV.9.1 of Dunford and Schwartz (1958), $\mathcal{C}(\nu_i)$ is weakly sequentially compact.[3] Therefore, replacing " $\frac{1}{2^i}$ " by " $\frac{1}{m_n}$ " at the bottom of p. 307, Theorem IV.9.2 of Dunford and Schwartz (1958) guarantees the existence of the desired $\lambda_i \in \mathcal{C}(\nu_i)$. $\square$

Claim 2. *$\mathcal{C}(\nu_i)$ consists of nonatomic probability measures.*

Assume that there exists an atom for λ_i, i.e. a set $A \in \Sigma$ with $\lambda_i(A) > 0$ and such that, for any $\Sigma \ni B \subseteq A$, either $\lambda_i(B) = 0$ or $\lambda_i(B) = \lambda_i(A)$. Since $\lambda_i(A) > 0$, then $\nu_i(A) > 0$. Let $\Sigma \ni B \subseteq A$. Either $\lambda_i(B) = 0$, then $\mu(B) = 0$ for every $\mu \in \mathcal{C}(\nu_i)$ and $\nu_i(B) = 0$; or $\lambda_i(B) = \lambda_i(A)$, then $\lambda_i(A - B) = 0$ and $\mu(A - B) = 0$ for every $\mu \in \mathcal{C}(\nu_i)$, so $\mu(B) = \mu(A)$ for every $\mu \in \mathcal{C}(\nu_i)$, whence $\nu_i(B) = \nu_i(A)$. Then A is an atom for ν_i, a contradiction. Therefore λ_i is nonatomic. Next we show that this implies the nonatomicity of all elements μ in $\mathcal{C}(\nu_i)$. Suppose, *per contra,* that there exists a μ in $\mathcal{C}(\nu_i)$ having an atom A. Since $\mu(A) > 0$, we have $\lambda_i(A) > 0$. Let $\{A_1, B_1\}$ be a partition of A such that $\lambda_i(A_1) = \lambda_i(B_1) = \frac{1}{2}\lambda_i(A)$. It must be either $\mu(A_1) = \mu(A)$ or $\mu(B_1) = \mu(A)$. Without loss of generality, assume $\mu(A_1) = \mu(A)$, A_1 is an atom for μ. Let $\{A_2, B_2\}$ be a partition of A_1 such that $\lambda_i(A_2) = \lambda_i(B_2) = \frac{1}{2}\lambda_i(A_1) = \frac{1}{2^2}\lambda_i(A)$. It must be either $\mu(A_2) = \mu(A_1) = \mu(A)$ or $\mu(B_2) = \mu(A_1) = \mu(A)$. Without loss of generality, assume $\mu(A_2) = \mu(A)$. Proceeding in this way, we can construct a decreasing sequence $\{A_k\} \subseteq \Sigma$ such that $\lambda_i(A_k) = \frac{1}{2^k}\lambda_i(A)$ and $\mu(A_k) = \mu(A)$ for all $k \geq 1$. Hence, $\lambda_i\left(\bigcap_{k=1}^{\infty} A_k\right) = 0$ and $\mu\left(\bigcap_{k=1}^{\infty} A_k\right) = \mu(A) > 0$, a contradiction. $\square$

Corollary 1.1 of Dubins and Spanier (1961) guarantees the existence of a partition $\{A_1, A_2, ..., A_n\}$ of S in Σ such that

$$\nu_i(A_i) \geq \lambda_i(A_i) \geq \alpha_i$$

for each $i = 1, 2, ..., n$.

If $\nu_j \neq \nu_k$, it must be $\mathcal{C}(\nu_j) \neq \mathcal{C}(\nu_k)$. Choose $\mu_j \in \mathcal{C}(\nu_j)$ and $\mu_k \in \mathcal{C}(\nu_k)$ such that $\mu_j \neq \mu_k$ and $\mu_i \in \mathcal{C}(\nu_i)$ arbitrarily, if $i \neq j, k$. If $\alpha_1, \alpha_2, ..., \alpha_n > 0$, by Corollary 1.2 of Dubins and Spanier (1961), there exists a partition $\{A_1, A_2, ..., A_n\}$ of S in Σ such that

$$\nu_i(A_i) \geq \mu_i(A_i) > \alpha_i$$

[3] That is, for any sequence $\{\mu_n\} \subseteq \mathcal{C}(\nu_i)$ there exists a subsequence $\{\mu_{n_k}\}$, such that $\mu_{n_k}(A)$ converges for all $A \in \Sigma$.

for each $i = 1, 2, ..., n$. Q.E.D.

As the elements $A_1, ..., A_n$ of the partition in Theorem 1 can be nastily shaped, it is important to know whether it is possible to guarantee to each participant a true slice of the cake rather than a bunch of crumbs. This problem is relevant in territorial disputes where, for example, n farmers have to partition a land bordering each of their properties. Hill (1983) solves the problem for evaluations represented by nonatomic probability measures; the next theorem extends his result to the case of nonatomic concave capacities. The proof, which is similar to that of Theorem 1, is omitted.

Theorem 2 (Fair Border) *Let $L, F_1, F_2, ..., F_n$ be open connected subsets in $\mathbb{R}^2$ with F_i adjacent to L for all $i = 1, 2, ..., n$.*[4] *If $\nu_1, \nu_2, ..., \nu_n$ are nonatomic concave (Borel) capacities on L, and $\alpha_1, \alpha_2, ..., \alpha_n \geq 0$ with $\sum_{i=1}^n \alpha_i = 1$. Then there exist disjoint open connected subsets $A_1, A_2, ..., A_n$ of L such that*

- *A_i is adjacent to F_i for all $i = 1, 2, ..., n$,*
- *$\nu_i(A_i) \geq \alpha_i$ for all $i = 1, 2, ..., n$, and*
- *$\overline{\bigcup_{i=1}^n A_i} \supseteq L$.*

Moreover, if $\nu_j \neq \nu_k$ for some $j \neq k$ and $\alpha_1, \alpha_2, ..., \alpha_n > 0$, then $A_1, A_2, ..., A_n$ can be chosen to satisfy

$$\nu_i(A_i) > \alpha_i$$

for each $i = 1, 2, ..., n$.

Like the original result, this theorem holds for subsets of $\mathbb{R}^k$, and, dropping all adjacency requirements, it yields the existence of a fair division of a connected cake into connected slices.

Notice that the hypotheses of both theorems can be weakened by requiring instead of concavity the condition:

(d') $\nu(A) = \max_{\mu \in \mathcal{C}(\nu)} \mu(A)$ for all $A \in \Sigma$,

called *exactness* (see Claim 0 in the proof of Theorem 1). As shown in Schmeidler (1972), exactness is a condition weaker than concavity and stronger than *subadditivity*:

(d") $\nu(C \cup A) \leq \nu(C) + \nu(A)$ for all disjoint $A, C \in \Sigma$,

which says that the individual is ***more*** *willing to eat slice A when she had no pizza at all* ***rather than*** *when she already ate a slice C.*[5] We are not sure whether subadditivity is sufficient to yield our existence results.

[4] Open connected subsets A and B of $\mathbb{R}^2$ are *adjacent* if $\partial A \cap \partial B$ contains an open arc γ (homeomorphic image of $(0, 1)$) such that $A \cup B \cup \gamma$ is open and connected.

[5] Notice that condition (d") can be obtained by setting $B = \emptyset$ in Eq. (1) in the Introduction.

3 Banach-Knaster Revisited

In this section we observe that two famous constructive solutions to the fair division problem, originally stated with additivity as one of the assumptions, can be obtained in the same way with subadditivity in place of additivity.

The following famous result, due to Banach and Knaster, constructively yields a division of a cake among n individuals such that any individual evaluates the slice she has received at least $\frac{1}{n}$.

> "...The partners being ranged A, B, C, . . . , N, A cuts from the cake an arbitrary part. B has now the right, but is not obliged, to diminish the slice cut off. Whatever he does, C has the right (without obligation) to diminish still the already diminished (or not diminished) slice, and so on up to N. The rule obliges the "last diminisher" to take as his part the slice he was the last to touch. This partner being thus disposed of, the remaining n-1 persons start the same game with the remainder of the cake...The hypotheses underlying the procedure described are: (1) the ideal shares due to the partners are not contested by any of them; (2) the object is continuous, so as to make it possible to cut off it a slice of value pV, V being the value of the whole and p any fraction between 0 and 1; the same applies to every part which can be cut off the object; (3) the sum of values of parts is equal to the value of the whole; the same applies to every part..." Steinhaus (1948).

Reading the description, it is easy to realize that the method still works if one replaces (3) with:

(3') the sum of values of parts is *not smaller than* the value of the whole; the same applies to every part.

This is simply a restatement of subadditivity. Notice that, in the cited procedure, partners' assignments change from no cake to a piece of cake, a situation where subadditivity yields the same behavioral information as concavity in terms of decreasing marginal evaluation.

The disadvantage of this method is that the cake may well end up in crumbs when the last participants choose their shares. To overcome it, Dubins and Spanier (1961) refined the technique, suggesting the Sliding Knife procedure.

> "...A knife is slowly moved at constant speed parallel to itself over the top of the cake. At each instant the knife is poised so that it could cut a unique slice of the cake. As times goes by the potential slice increases monotonely from nothing until it becomes the entire cake. The first person to indicate satisfaction with the slice then determined by the position of the knife receives that slice and is eliminated from further distribution of the cake. (If two or more participants simultaneously indicate satisfaction with the slice, it is given to any one of them.) The process is repeated with the other n-1 participants and with what remains of the cake...The method described above is equally applicable for the division of any object provided only that (1) the value assigned by any participant to any part of the

object equals the sum of the values of the subparts when the part is subdivided into any finite number of subparts; and (2) the value to each participant of the potential slice varies in a continuous fashion as the knife is moved over the object..."

Again, the technique does not require the full power of (1), but the following restatement of subadditivity is enough:

(1') the value assigned by any participant to any part of the object is *not greater than* the sum of the values of the subparts when the part is subdivided into any finite number of subparts.

We conclude by observing that, when $\alpha_1, \alpha_2, ..., \alpha_n$ is a n-tuple of nonnegative rational numbers adding up to 1, for additive evaluations the above procedures can be easily adapted to yield a division in which the ith participant receives at least α_i. Barbanel (1995) suggested a method that yields (using the same assumptions as Banach and Knaster) such a partition even if the α_is are not rational. The extension of such results to subadditive or concave evaluations is the object of future research.

4 Concluding Remarks

1. In some situations it is possible to give natural behavioral conditions on the individuals' preferences that guarantee the existence of unique nonatomic exact (or concave) capacities representing them. In other words it is possible to give fully behavioral versions of Theorems 1 and 2. Consider the following bucolic example. In a village, n farmers share a common orchard and each farmer i is entitled to receive a fraction α_i of the whole production. The farmers suspect each other of stealing fruits at night and argue over who should work in the orchard. The mayor of the village would like to assign to each of them a single piece of land so to solve all the disputes.

For all pairwise disjoint subsets $B_1, B_2, ..., B_m$ of S and for all $\beta_1, \beta_2, ..., \beta_m \in [0,1]$, let $\beta_1 B_1 + \beta_2 B_2 + ... + \beta_m B_m$ be the entitlement to receive a fraction β_j of the production of lot B_j. All these entitlements are represented by the (convex) set of measurable simple functions on S taking values in $[0,1]$. By using standard techniques (see, e.g., Schmeidler, 1989, Gilboa and Schmeidler, 1989, and Chateauneuf, Maccheroni, Marinacci, and Tallon, 2002) it is easy to come up with axioms on the farmers' preferences $\succsim_i$ that deliver the existence of unique exact (or concave) capacities ν_i representing their evaluations of the lots.

2. Both the Cake Cutting and the Fair Border theorems rely on the Lyapunov Theorem; that is, on the fact that the range $\{(\mu_1(A), \mu_2(A), ..., \mu_n(A)) : A \in \Sigma\}$ of a vector $(\mu_1, \mu_2, ..., \mu_n)$ of nonatomic probability measures is a convex subset of $\mathbb{R}^n$. It is thus natural to wonder whether a similar result holds for a vector of nonatomic concave capacities. Unfortunately, this is not the case: it suffices to consider the vector $(\mu, \sqrt{\mu})$, where μ is the Borel measure on $[0,1]$.

3. Berliant, Dunz, and Thomson (1992) proved that a result analogous to Theorem 2 holds for the following class of set functions defined on the Borel σ-algebra $\mathcal{B}$ of an open subset L of $\mathbb{R}^k$:

> "...Let m be Lebesgue measure on $\mathbb{R}^k$...The function $u_i : \mathcal{B} \to \mathbb{R}_+$ is *concave* if there exists a function $h_i : \{(x, B) \in L \times \mathcal{B} : x \in B\} \to \mathbb{R}_+$ such that
> (i) for all $B \in \mathcal{B}$, $h_i(\cdot, B)$ is integrable,
> (ii) for all $B, B' \in \mathcal{B}$ with $B' \subseteq B$, for all $x \in B'$, $h_i(x, B') \geq h_i(x, B)$, and
> (iii) for all $B \in \mathcal{B}$, $u_i(B) = \int_B h_i(x, B)\, dm(x)$..."

Setting $f_i(x) = h_i(x, L)$ for each i, it is easy to see that (i)-(iii) are stronger than the following condition:

> (iv) There exists an integrable function $f_i : L \to \mathbb{R}_+$ such that $u_i(B) \geq \int_B f_i(x)\, dm(x)$ for all $B \in \mathcal{B}$ and equality holds if $B = L$.

On the other hand, normalizing $u_i(L) = 1$ for each i, it is easy to show that condition (iv) yields Theorem 2. In fact, the set function $\mu_i(B) = \int_B f_i(x)\, dm(x)$ is a nonatomic probability measure on $\mathcal{B}$ such that $u_i \geq \mu_i$; then, by Theorem 2 of Hill (1983), the desired fair division exists.

It is important to observe that, differently from our assumptions, both (i)-(iii) and (iv) invoke the existence of the "extraneous objects" h_is and f_is.

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